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CR SUBMANIFOLDS OF A KAEHLER MANIFOLD. II

BY

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ABSTRACT. The differential geometry of *CR* submanifolds of a Kaehler manifold is studied. Theorems on parallel normal sections and on a special type of flatness of the normal connection on a *CR* submanifold are obtained. Also, the nonexistence of totally umbilical proper *CR* submanifolds in an elliptic or hyperbolic complex space is proven.

1. Introduction and basic formulas. A study of the differential geometry of *CR* submanifolds of a Kaehler manifold has been initiated in [2]. Some problems, mostly concerning integrability conditions on such submanifolds have been investigated by S. Ishihara and K. Yano [6]. Also, results on the general theory of Cauchy-Riemann manifolds have been obtained by A. Andreotti and C. D. Hill [1], R. Nirenberg and R. O. Wells [8], V. Oproiu [7], and others.

The purpose of the present paper is to study further *CR* submanifolds by the method which has been used in [2]. First, in §2 we state theorems on parallel normal sections on a *CR* submanifold and some lemmas for later use. A special type of flatness for the normal connection of a *CR* submanifold is studied in §3. Finally, in §4 we give some characterizations for a class of *CR* submanifolds, and the nonexistence of totally umbilical *CR* submanifolds in an elliptic or hyperbolic complex space is proven. In this paragraph we give the basic definitions and formulas.

Let $\tilde{M}$ be a Kaehler manifold of complex dimension n and M be a submanifold of $\tilde{M}$ of real dimension m. The submanifold M is supposed to be endowed with two complementary orthogonal distributions D and $D^{\perp}$ of real dimensions $2p$ and q respectively. The first one is called the *horizontal distribution* and it is invariant by the almost complex structure J on $\tilde{M}$: that is, $J(D_x) = D_x$ for each $x \in M$. The second one is called the *vertical distribution* and it is anti-invariant by J; that is, $J(D_x^{\perp}) \subset \nu_x$ where ν_x is the normal space to M at x.

The submanifold M endowed with the pair of distributions $(D, D^{\perp})$ is called a *CR submanifold* of $\tilde{M}$ [2]. It is easily seen that each real hypersurface of $\tilde{M}$ is a *CR* submanifold.

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REMARK. Of course, the definition of a CR submanifold might be considered for submanifolds of an almost Hermitian manifold [3], but we are concerned, in this paper, only with problems on CR submanifolds of a Kaehler manifold.

The Kaehlerian metric on $\tilde{M}$ and the Riemannian metric induced by it on M will be denoted by the same symbol g. The Kaehlerian connection on $\tilde{M}$ is denoted by $\tilde{\nabla}$, the Levi-Civita connection on M is denoted by ∇ and by $\nabla^{\perp}$ is denoted the linear connection induced by $\tilde{\nabla}$ on the normal bundle ν. Then the equations of Gauss and Weingarten are

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.1}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{1.2}$$

for any vector fields X, Y tangent to M and any vector field N in the normal bundle ν. We put

$$JN = BN + CN, \tag{1.3}$$

where BN and CN are the vertical and the normal component of JN respectively.

The second fundamental form h of M verifies

$$g(h(X, Y), N) = g(A_N X, Y). \tag{1.4}$$

Our study is made on the domain of a local chart of coordinates on M. For any vector bundle $S \to M$ on M, the module of its local sections is denoted by $\mathbb{S}(S)$, without specifying the local chart. Also, throughout the paper, all the manifolds and mappings are supposed to be differentiable of class C^{∞}.

The curvature tensor of type (1, 3) of M is given by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}.$$

Also, we shall use the curvature tensor of type (0, 4) given by

$$R(X, Y; Z, W) = g(R(X, Y)Z, W).$$

Let $\{E_1, \ldots, E_m\}$ be a local orthonormal frame on M. Then, the Ricci tensor S of M is given by

$$S(X, Y) = \sum_{i=1}^{m} \{R(E_i, X; Y, E_i)\}$$

where X, Y are vector fields tangent to M.

Of course, similar formulas are given for the curvature tensor $\tilde{R}$ and the Ricci tensor $\tilde{S}$ of $\tilde{M}$.

Denote by $R^{\perp}$ the curvature tensor of the normal connection $\nabla^{\perp}$. A local normal vector field $N \neq 0$ is said to be a *D-parallel normal section* if $\nabla^{\perp}_X N = 0$ for each $X \in \mathbb{S}(D)$.

The equations of Gauss, Codazzi and Ricci are given respectively by [4, p. 46]

$$\tilde{R}(X, Y; Z, W) = R(X, Y; Z, W) + g(h(X, Z), h(Y, W)) - g(h(Y, Z), h(X, W)), \tag{1.5}$$

$$[\tilde{R}(X, Y)Z]^{\perp} = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \tag{1.6}$$

$$\tilde{R}(X, Y; N, \bar{N}) = R^{\perp}(X, Y; N, \bar{N}) - g([A_N, A_{\bar{N}}](X), Y) \tag{1.7}$$

for any vector fields X, Y, Z, W tangent to M and normal vector fields N, $\bar{N}$. The left-hand side of (1.6) is the normal component of $\tilde{R}(X, Y)Z$ and $\bar{\nabla}_X h$ from the right-hand side is given by

$$(\bar{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{1.8}$$

Now, suppose the distributions D and $D^{\perp}$ are given by the projectors P and Q respectively. Then, since $\tilde{\nabla}$ is a Kaehlerian connection, from the equations of Gauss and Weingarten, by comparing horizontal, vertical and normal parts, we obtain respectively

$$P(\nabla_X JPY) - P(A_{JQY}X) = JP(\nabla_X Y), \tag{1.9}$$

$$Q(\nabla_X JPY) - Q(A_{JQY}X) = Bh(X, Y), \tag{1.10}$$

$$h(X, JPY) + \nabla^{\perp}_X JQY = JQ(\nabla_X Y) + Ch(X, Y) \tag{1.11}$$

for any vector fields X, Y tangent to M.

2. D-parallel normal sections on a CR submanifold. Let M be a CR submanifold of a Kaehler manifold $\tilde{M}$. As it was supposed in §1, $JD^{\perp}$ is a vector subbundle of the normal bundle ν. The complementary orthogonal vector bundle of $JD^{\perp}$ in ν will be denoted by μ. It is easily seen that μ is invariant by the almost complex structure J.

The CR submanifold M is said to be *mixed totally geodesic* if $h(X, Y) = 0$ for any vector fields $X \in \mathcal{S}(D)$ and $Y \in \mathcal{S}(D^{\perp})$.

LEMMA 2.1. *A CR submanifold M of a Kaehler manifold $\tilde{M}$ is mixed totally geodesic if and only if $A_N X \in \mathcal{S}(D)$ for each $X \in \mathcal{S}(D)$ and $N \in \mathcal{S}(\nu)$.*

PROOF. If M is mixed totally geodesic, then by using (1.4) we have $g(A_N X, Y) = 0$ for each $X \in \mathcal{S}(D)$, $Y \in \mathcal{S}(D^{\perp})$ and $N \in \mathcal{S}(\nu)$. This implies $A_N X \in \mathcal{S}(D)$.

Conversely, suppose $A_N X \in \mathcal{S}(D)$ for any $X \in \mathcal{S}(D)$ and $N \in \mathcal{S}(\nu)$. Let $\{N_1, \ldots, N_{2n-m}\}$ be a local orthonormal field of frames on ν. Then we have

$$0 = g(A_{N_\alpha} X, Y) = g(h(X, Y), N_\alpha), \qquad 1 \leq \alpha \leq 2n - m.$$

Since $h(X, Y) \in \mathcal{S}(\nu)$, from the relations above we have $h(X, Y) = 0$. Therefore M is mixed totally geodesic.

LEMMA 2.2. *Let M be a mixed totally geodesic CR submanifold of a Kaehler manifold $\tilde{M}$. Then we have*

$$A_{JN}X = JA_N X \tag{2.1}$$

for any $X \in \mathcal{S}(D)$ and $N \in \mathcal{S}(\mu)$.

PROOF. Since $\tilde{\nabla}$ is a Kaehlerian connection, from the equation of Weingarten we get

$$JA_N X - J\nabla_X^{\perp} N = A_{JN}X - \nabla_X^{\perp} JN. \tag{2.2}$$

From Lemma 2.1 the first terms from both sides of (2.2) belong to $\mathcal{S}(D)$. On the other hand, $\nabla_X^{\perp} JN \in \mathcal{S}(\nu)$ and $J\nabla_X^{\perp} N \in \mathcal{S}(D^{\perp} \oplus \mu)$. Hence (2.1) follows from (2.2) by comparing the horizontal parts of both sides.

Also, from (2.2) we obtain

COROLLARY 2.1. *If M is a mixed totally geodesic CR submanifold of a Kaehler manifold M, then we have*

$$J\nabla_X^{\perp} N = \nabla_X^{\perp} JN \tag{2.3}$$

and

$$\nabla_X^{\perp} N \in \mathcal{S}(\mu) \tag{2.4}$$

for any vector fields $X \in \mathcal{S}(D)$ and $N \in \mathcal{S}(\mu)$.

If the horizontal distribution is involutive, then M will be called a *foliate CR submanifold* of $\tilde{M}$. We have proved [2]

PROPOSITION 2.1. *A CR submanifold M of a Kaehler manifold $\tilde{M}$ is foliate if and only if the second fundamental form h satisfies*

$$h(X, JY) = h(JX, Y) \tag{2.5}$$

for each $X, Y \in \mathcal{S}(D)$.

Now we prove

LEMMA 2.3. *Let M be a foliate CR submanifold of a Kaehler manifold $\tilde{M}$. If M is mixed totally geodesic, then we have*

$$JA_N X = -A_N JX \tag{2.6}$$

for any $X \in \mathcal{S}(D)$ and $N \in \mathcal{S}(\nu)$.

PROOF. From (1.4) and (2.5) we have

$$g(JA_N X, Y) = -g(h(X, JY), N) = -g(h(JX, Y), N) = -g(A_N JX, Y) \tag{2.7}$$

for any vector fields $X, Y \in \mathcal{S}(D)$ and $N \in \mathcal{S}(\nu)$. Thus (2.6) follows from (2.7) by using Lemma 2.1.

The *holomorphic bisectional curvature* for the pair of vector fields (X, Y) on $\tilde{M}$ is given by

$$H(X, Y) = \tilde{R}(X, JX; JY, Y)/g(X, X)g(Y, Y).$$

Now we can state the following theorem which has been proved by B. Y. Chen and H. S. Lue for complex submanifolds of a Kaehler manifold **[5]**.

THEOREM 2.1. *Let M be a mixed totally geodesic foliate CR submanifold of a Kaehler manifold $\tilde{M}$. If there exists a unit vector field $X \in \mathbb{S}(D)$ such that for all normal sections $N \in \mathbb{S}(\mu)$, the holomorphic bisectional curvatures $H(X, N)$ are positive, then the normal subbundle μ does not admit D-parallel section.*

PROOF. Suppose N is a parallel section of μ. Then $R^{\perp}(X, Y)N = 0$ for each $X, Y \in \mathbb{S}(D)$. Hence, by using the equation of Ricci we have

$$\tilde{R}(X, Y; N, JN) = -g([A_N, A_{JN}](X), Y). \tag{2.8}$$

Next, by (2.1) and (2.6) the second-hand side of (2.8) becomes $2g(JA_N^2X, Y)$.

On the other hand, the hypothesis on the holomorphic bisectional curvatures and (2.8) imply

$$0 > \tilde{R}(X, JX; N, JN) = 2g(A_N^2X, X).$$

This is clearly impossible, since g is positive definite.

Now we give a characterization for the parallel normal sections which belong to the normal subbundle $JD^{\perp}$.

THEOREM 2.2. *Let M be a mixed totally geodesic CR submanifold of a Kaehler manifold $\tilde{M}$. Then the normal section $N \in \mathbb{S}(JD^{\perp})$ is D-parallel if and only if $\nabla_X JN \in \mathbb{S}(D)$ for each vector field $X \in \mathbb{S}(D)$.*

PROOF. Let $Y \in \mathbb{S}(D^{\perp})$ be such that $JY = N$. Then from (1.9) we have

$$PA_NX = -JP(\nabla_X Y)$$

for each $X \in \mathbb{S}(D)$. From Lemma 2.1 we have $A_NX \in \mathbb{S}(D)$. Hence the relation above becomes

$$A_NX = -JP(\nabla_X Y). \tag{2.9}$$

Now, from the equations of Gauss and Weingarten, by using $\tilde{\nabla}J = 0$ and (2.9) we obtain

$$\nabla_X^{\perp}N = JQ(\nabla_X Y) + Ch(X, Y). \tag{2.10}$$

Since M is mixed geodesic, from (2.10) we have

$$\nabla_X^{\perp}N = -JQ(\nabla_X JN), \tag{2.11}$$

which clearly proves the theorem.

3. The normal connection of a *CR* submanifold. The normal connection $\nabla^{\perp}$ of the CR submanifold M is called **(D, μ)**-*flat*, if the restriction of the curvature tensor $R^{\perp}$ of $\nabla^{\perp}$ to $\mathcal{S}(D) \times \mathcal{S}(D) \times \mathcal{S}(\mu)$ vanishes, i.e.

$$R^{\perp}(X, Y)N = [\nabla_X^{\perp}, \nabla_Y^{\perp}](N) - \nabla_{[X, Y]}^{\perp}N = 0 \tag{3.1}$$

for each $X, Y \in \mathcal{S}(D)$ and $N \in \mathcal{S}(\mu)$.

The theorem which follows establishes a link between the (D, μ)-flatness of the normal connection and the existence of D-parallel normal sections from μ. Suppose the fibre of the vector bundle μ at each point of M is of complex dimension $s \geqslant 1$.

THEOREM 3.1. *Let M be a mixed totally geodesic foliate CR submanifold of a Kaehler manifold $\tilde{M}$. Then, the normal connection is (D, μ)-flat if and only if there exist locally $2s$ mutually orthogonal unit normal vector fields*[1] *$N_a \in \mathcal{S}(\mu)$ such that each of the N_a is D-parallel in the normal subbundle μ.*

PROOF. Suppose the normal connection $\nabla^{\perp}$ is (D, μ)-flat. Take a point $x \in M$ and denote by M_x the maximal integral manifold of D passing through x. For each vector field X on M_x put

$$\nabla_X^{\perp}N_a = \omega_a^b(X)N_b.$$

This is possible by virtue of (2.4). Then, from (3.1) we get

$$d\omega_a^b = -\omega_a^c \wedge \omega_c^b; \qquad \omega_a^b + \omega_b^a = 0.$$

Then we can find an $(s \times s)$-nonsingular matrix $A = (A_a^b)$ of functions such that $dA_a^b = -A_a^c\omega_c^b$. Now, we take $N_a' = A_a^bN_b$. Of course, N_a' are also $2s$ mutually orthogonal unit normal vector fields from μ. Moreover, if we put $\nabla_X^{\perp}N_a' = \omega_a'^b(X)N_b'$, then we have

$$\omega_a^cA_c^b = dA_a^b + A_a^c\omega_c^b = 0,$$

which implies $\omega_a'^c = 0$ for all indices a and c. Hence N_a' are $2s$ D-parallel orthonormal vector fields from the normal subbundle μ.

Conversely, suppose that there exist locally $2s$ mutually orthogonal unit normal vector fields N_a such that each N_a is D-parallel in the normal subbundle μ. Let N be a certain normal vector field from μ. Since $R^{\perp}(X, Y)N_a = 0$, by a simple computation we get $R^{\perp}(X, Y)N = 0$ for any $X, Y \in \mathcal{S}(D)$. Thus the normal connection is (D, μ)-flat and the proof is complete.

Let $\{F_1, \ldots, F_q\}$ be a local field of frames on the vertical distribution. The CR submanifold M is said to be $D^{\perp}$*-minimal*, if the second fundamental form h of M satisfies

$$\sum_{k=1}^{q} \{h(F_k, F_k)\} = 0. \tag{3.2}$$

[1]The indices $a, b, c, \ldots$ run over the range $1, 2, \ldots, 2s$, and we use Einstein's convention on summing indices.

The definition above does not depend on the local field of frames $\{F_1, \ldots, F_q\}$. In fact, the second fundamental form h can be written as $h(X, Y) = h^\alpha(X, Y)N_\alpha$, $\alpha = 1, \ldots, 2n - m$, where $\{N_1, \ldots, N_{2n-m}\}$ is a local field of orthonormal frames in the normal bundle. Denote by $h^\alpha_{D^\perp}$ the restriction of h^α to $\mathbb{S}(D^\perp) \times \mathbb{S}(D^\perp)$. Then, the condition (3.2) is equivalent to

$$\operatorname{trace} h^\alpha_D = 0, \text{ for each } \alpha = 1, \ldots, 2n - m. \tag{3.3}$$

Now it is easily seen from (3.3) that the definition of a $D^\perp$-minimal CR submanifold does not depend on the basis. In the same way, we say that M is *D-minimal,* if the second fundamental form satisfies

$$\sum_{i=1}^{2p} \{h(E_i, E_i)\} = 0, \tag{3.4}$$

where $\{E_1, \ldots, E_{2p}\}$ is a local field of orthonormal frames on the horizontal distribution.

THEOREM 3.2. *Let M be a mixed totally geodesic foliate CR submanifold of a Kaehler manifold $\tilde{M}$. If the normal connection is (D, μ)-flat and M is $D^\perp$-minimal, then the Ricci tensors S and $\tilde{S}$ of M and respectively $\tilde{M}$ satisfy the relation*

$$S(X, Y) = \tilde{S}(X, Y) - \sum_{k=1}^{q} \{g(P\nabla_X F_k, P\nabla_Y F_k) + \tilde{R}(JF_k, X; Y, JF_k)\}, \tag{3.5}$$

for each $X, Y \in \mathbb{S}(D)$ and for any local field of orthonormal frames $\{F_1, \ldots, F_q\}$ on the vertical distribution.

PROOF. Let

$$\{E_1, \ldots, E_{2p}, F_1, \ldots, F_q, JF_1, \ldots, JF_q, N_1, \ldots, N_s, JN_1, \ldots, JN_s\}$$

be a local field of orthonormal frames of $\tilde{M}$ such that $\{E_1, \ldots, E_{2p}\}$, $\{F_1, \ldots, F_q\}$ and $\{N_1, \ldots, N_s, JN_1, \ldots, JN_s\}$ are local fields of frames of the horizontal distribution, vertical distribution and of the normal subbundle μ respectively. Moreover, the vector fields $\{N_1, \ldots, N_s\}$ are taken as D-parallel vector fields from those whose existence has been proven by Theorem 3.1. From (2.3) we see that $\{JN_1, \ldots, JN_s\}$ are also D-parallel normal sections. Next, by using the definition of Ricci tensor and the equation of Gauss, we get

$$S(X, Y) = \tilde{S}(X, Y) + g\left(h(X, Y), \sum_{i=1}^{2p} \{h(E_i, E_i)\} + \sum_{k=1}^{q} \{h(F_k, F_k)\}\right)$$
$$- \sum_{i=1}^{2p} \{ g(h(E_i, X), h(E_i, Y))\}$$
$$- \sum_{k=1}^{q} \{ g(h(F_k, X), h(F_k, Y)) + \tilde{R}(JF_k, X; Y, JF_k)\}$$
$$- \sum_{\alpha=1}^{s} \{\tilde{R}(N_\alpha, X; Y, N_\alpha) + R(JN_\alpha, X; Y, JN_\alpha)\} \tag{3.6}$$

for each $X, Y \in \mathfrak{S}(D)$.

Since the horizontal distribution is involutive, M is D-minimal [2]. On the other hand, M is supposed to be $D^\perp$-minimal. Hence the second term from the right-hand side of (3.6) vanishes.

Now, from (1.4) and by using Lemmas 2.1 and 2.2, we get

$$\sum_{i=1}^{2p} \{ g(h(E_i, X), h(E_i, Y))\}$$
$$= \sum_{k=1}^{q} \{ g(A_{JF_k}X, A_{JF_k}Y)\} + 2 \sum_{\alpha=1}^{s} \{ g(A_{N_\alpha}X, A_{N_\alpha}Y)\}.$$

We make use again of the equations of Gauss and Weingarten and obtain $A_{JF_k}X = -JP\nabla_X F_k$, which transforms the relation above into

$$\sum_{i=1}^{2p} \{ g(h(E_i, X), h(E_i, Y))\}$$
$$= \sum_{k=1}^{q} \{ g(P\nabla_X F_k, P\nabla_X F_k)\} + 2 \sum_{\alpha=1}^{s} \{ g(A_{N_\alpha}X, A_{N_\alpha}Y)\}. \tag{3.7}$$

For the last sum of the right-hand side of (3.6) we have the following evaluations:

$$\sum_{\alpha=1}^{s} \{\tilde{R}(N_\alpha, X; Y, N_\alpha) + \tilde{R}(JN_\alpha, X; Y, JN_\alpha)\}$$
$$= - \sum_{\alpha=1}^{s} \{\tilde{R}(X, JY; N_\alpha, JN_\alpha)\} = -2 \sum_{\alpha=1}^{s} \{ g(A_{N_\alpha}X, A_{N_\alpha}Y)\}. \tag{3.8}$$

Finally, we see that (3.5) follows from (3.6) by using (3.7), (3.8) and the fact that M is mixed totally geodesic. Thus the proof is done.

The vertical distribution is said to be *parallel along the horizontal distribution* if we have $\nabla_X Y \in \mathfrak{S}(D)^\perp$ for each $X \in \mathfrak{S}(D)$ and $Y \in \mathfrak{S}(D^\perp)$.

COROLLARY 3.1. *Let M be a CR submanifold of the Kaehler manifold $\tilde{M}$ under the conditions of Theorem* 3.2. *Suppose the following two conditions are satisfied*:

(1) *The vertical distribution is parallel along the horizontal distribution.*

(2) *The curvature tensor R satisfies $R(Z, X)Y \in \mathbb{S}(D)$, for any $X, Y \in \mathbb{S}(D)$ and $Z \in \mathbb{S}(D^{\perp})$.*

Then, the Ricci tensors S and $\tilde{S}$ satisfy

$$S(X, Y) = \tilde{S}(X, Y), \tag{3.9}$$

for any $X, Y \in \mathbb{S}(D)$.

PROOF. From condition (1). of the corollary we see that $P\nabla_X F_k = 0$ for each $X \in \mathbb{S}(D)$ and $1 \leqslant k \leqslant q$. Hence the first q terms from the sum of the right-hand side of (3.5) vanish. It is known that the curvature tensor $\tilde{R}$ of the Kaehler manifold $\tilde{M}$ satisfies

$$\tilde{R}(JX, JY) = \tilde{R}(X, Y) \tag{3.10}$$

and

$$\tilde{R}(X, Y)JZ = J\tilde{R}(X, Y)Z, \tag{3.11}$$

for any vector fields X, Y, Z tangent to $\tilde{M}$. Then, by using (3.10) and (3.11) we obtain

$$\tilde{R}(JF_k, X; Y, JF_k) = \tilde{R}(F_k, JX; JY, F_k), \tag{3.12}$$

for any vector fields $X, Y \in \mathbb{S}(D)$. Now, from (1.5), taking into account that M is $D^{\perp}$-minimal and mixed totally geodesic, we get

$$\sum_{k=1}^{q} \{\tilde{R}(F_k, JX; JY, F_k)\} = \sum_{k=1}^{q} \{R(F_k, JX; JY, F_k)\}. \tag{3.13}$$

Condition (2) of the corollary implies

$$R(F_k, JX; JY, F_k) = 0, \quad k = 1, \ldots, q. \tag{3.14}$$

Finally, from (3.12), (3.13) and (3.14) we conclude that

$$\sum_{k=1}^{q} \{\tilde{R}(JF_k, X; Y, JF_k)\} = 0, \tag{3.15}$$

and the corollary follows from (3.5).

The Kaehler manifold $\tilde{M}$ is said to be an *Einstein space* if there exists a constant ρ such that $\tilde{S}(X, Y) = \rho g(X, Y)$ for all X, Y tangent to $\tilde{M}$.

Now we can prove

THEOREM 3.3. *Let $\tilde{M}$ be an Einstein-Kaehler manifold. If M is a CR submanifold of $\tilde{M}$ and satisfies the conditions from Corollary* 3.1 *then each maximal integral manifold of D is totally geodesic immersed in M if and only if it is an Einstein space with the same scalar curvature as $\tilde{M}$.*

PROOF. Let M^* be the maximal integral manifold of D passing through the point $x \in M$. Denote by S^* the Ricci tensor of M^* and by h^* the second fundamental form of M^* considered as a submanifold of M. Then, for each pair of local vector fields X, Y tangent to M^* we obtain

$$S(X, Y) = S^*(X, Y) + \sum_{i=1}^{2p} \{ g(h^*(E_i, X), h^*(E_i, Y)) \}$$
$$-g\left(h^*(X, Y), \sum_{i=1}^{2p} \{ h^*(E_i, E_i) \} \right) + \sum_{k=1}^{q} \{ g(R(F_k, X)Y, F_k) \}, \quad (3.16)$$

where $\{E_1, \ldots, E_{2p}, F_1, \ldots, F_q\}$ is a local orthonormal field of frames on M such that E_i, $1 \leqslant i \leqslant 2p$, are tangent to M and $F_k \in \mathfrak{S}(D^\perp)$. Denote by $\tilde{h}$ the second fundamental form of M^* as submanifold of $\tilde{M}$. Then we have

$$\tilde{h}(X, Y) = h^*(X, Y) + h(X, Y). \quad (3.17)$$

M^* is a Kaehler submanifold of $\tilde{M}$; hence it is minimal. On the other hand, M is D-minimal since it is foliate. Therefore, from (3.17) M^* is minimal as submanifold of M. Then by using Corollary 3.1, (3.16) becomes

$$\tilde{S}(X, Y) = S^*(X, Y) + \sum_{i=1}^{2p} \{ g(h^*(E_i, X), h^*(E_i, Y)) \}. \quad (3.18)$$

Now it is easily seen that (3.18) proves the theorem.

4. ***CR*** **submanifolds of a complex space form.** Let $\tilde{M}$ be a complex space form of constant holomorphic sectional curvature c. The curvature tensor of $\tilde{M}$ is given by

$$\tilde{R}(X, Y)Z = \tfrac{c}{4}\{ g(Y, Z)X - g(X, Z)Y + g(Z, JY)JX$$
$$- g(Z, JX)JY + 2g(X, JY)JZ \}, \quad (4.1)$$

where X, Y, Z are vector fields on $\tilde{M}$.

A *CR* submanifold M of $\tilde{M}$ is called (D, μ)*-totally geodesic*, if we have $A_N X = 0$ for each $X \in \mathfrak{S}(D)$ and $N \in \mathfrak{S}(\mu)$.

THEOREM 4.1. *Let M be a mixed totally geodesic foliate CR submanifold of a complex space form $\tilde{M}(c)$. If the normal connection is (D, μ)-flat, then $c \leqslant 0$. The equality holds good if and only if M is (D, μ)-totally geodesic.*

PROOF. From the equation of Ricci, taking into account the (D, μ)-flatness of the normal connection we obtain

$$g\left(\left[\tilde{R}(X, Y)N \right]^\perp, JN \right) = g(A_{JN}Y, A_N X) - g(A_{JN}X, A_N Y),$$

for each $N \in \mathfrak{S}(\mu)$ and $X, Y \in \mathfrak{S}(D)$. Next, by using (2.1) and (2.6) the relation above becomes

$$g\left(\left[\tilde{R}(X, Y)N\right]^{\perp}, JN\right) = -2g(A_N X, A_N JY). \tag{4.2}$$

On the other hand, from (4.1) we have

$$g\left(\left[\tilde{R}(X, Y)N\right]^{\perp}, JN\right) = \frac{c}{2} g(X, JY) g(N, N). \tag{4.3}$$

Take N as a unit section of the normal subbundle μ. Then from (4.2) and (4.3) we obtain

$$cg(X, Y) + 4g(A_N X, A_N Y) = 0. \tag{4.4}$$

Since g is a positive definite metric, from (4.4) we get $c \leqslant 0$. The last part of the theorem follows from (4.4).

The vertical distribution $D^{\perp}$ is said to be *flat* with respect to the Levi-Civita connection on M if we have $\nabla_X Y \in \mathbb{S}(D^{\perp})$ for any vector fields $X, Y \in \mathbb{S}(D^{\perp})$. Now, we can state

THEOREM 4.2. *Let M be a CR submanifold of real codimension $q \geqslant 2$ of a complex space form $\tilde{M}(c)$ of complex dimension $p + q$. If the vertical distribution of M is flat with respect to the Levi-Civita connection on M, and $[A_N, A_{\bar{N}}] = 0$ for any normal vectors N and $\bar{N}$, then the sectional curvature K_M of M satisfies*

$$K_M(X \wedge Y) = \frac{c}{4}, \tag{4.5}$$

for each pair of orthonormal vectors (X, Y) from $D^{\perp}$.

PROOF. From the Ricci equation, by using (4.1) and the assumption $[A_N, A_{\bar{N}}] = 0$, we get

$$\frac{c}{4}\left\{ g(BN, QY) g(B\bar{N}, QX) - g(BN, QX) g(B\bar{N}, QY) + 2g(PX, JPY) g(CN, \bar{N})\right\} = g\left(R^{\perp}(X, Y)N, \bar{N}\right),$$

for any vector fields X, Y tangent to M and $N, \bar{N}$ normal to M. Since $\dim \mu = 0$, we have $C = 0$. Thus, for the particular case when $X, Y \in \mathbb{S}(D^{\perp})$, the relation above becomes

$$\frac{c}{4}\left\{ g(JN, Y) g(J\bar{N}, X) - g(JN, X) g(J\bar{N}, Y)\right\} = g\left(R^{\perp}(X, Y)N, \bar{N}\right). \tag{4.6}$$

Now let $Z \in \mathbb{S}(D^{\perp})$ be such that $JZ = N$. The assumption on the codimension of M assures the existence of the vector field Z. Then from (1.1) and by using the fact that $D^{\perp}$ is flat with respect to ∇ we have $\nabla^{\perp}_X N = J\nabla_X Z$. This implies $R^{\perp}(X, Y)N = JR(X, Y)Z$. If we put $\bar{N} = JW$, $W \in \mathbb{S}(D^{\perp})$, then from (4.6) we get

$$g(R(X, Y)Z, W) = \tfrac{c}{4}\{g(Z, Y)g(W, X) - g(Z, X)g(W, Y)\}. \quad (4.7)$$

Now, (4.5) follows immediately from (4.7).

A CR submanifold M of a Kaehler manifold $\tilde{M}$ is called $D^{\perp}$-*totally geodesic* if $h(X, Y) = 0$ for any $X, Y \in \mathbb{S}(D^{\perp})$.

We have proved the following theorem [2]:

THEOREM 4.3. *If M is a $D^{\perp}$-minimal CR submanifold of a complex space form $\tilde{M}(c)$, then M is $D^{\perp}$-totally geodesic if and only if $K_M(X \wedge Y) = \frac{c}{4}$ for each pair of vector fields $X, Y \in \mathbb{S}(D^{\perp})$.*

Now, combining Theorems 4.2 and 4.3 we find

THEOREM 4.4. *If a CR submanifold M of a Kaehler manifold $\tilde{M}$ satisfies the conditions of Theorem 4.2, then M is $D^{\perp}$-totally geodesic.*

In what follows we shall study the existence of a certain class of CR submanifolds in a complex space form of non-null holomorphic sectional curvature.

A *proper CR submanifold M* of a Kaehler manifold $\tilde{M}$ is a CR submanifold with both distributions D and $D^{\perp}$ of non-null dimensions.

Also, M is said to be *totally umbilical* if there exists a normal vector field L, such that the second fundamental form h satisfies

$$h(X, Y) = g(X, Y)L, \quad (4.8)$$

for any vector fields X, Y tangent to M.

Now we can state

THEOREM 4.5. *There exist no totally umbilical proper CR submanifolds of an elliptic or hyperbolic complex space.*

PROOF. Suppose there exists a totally umbilical proper CR submanifold M of a complex space form $\tilde{M}$ $(c \neq 0)$. Let X and Y be two non-null vector fields from D and $D^{\perp}$ respectively. Then, for the normal part of $\tilde{R}(X, JX)Y$ we get

$$\left[\tilde{R}(X, JX)Y\right]^{\perp} = -\tfrac{c}{2}\, g(X, X)JY \neq 0. \quad (4.9)$$

On the other hand, since M is totally umbilical, from the Codazzi equation we have

$$\left[\tilde{R}(X, JX)Y\right]^{\perp} = g(JX, Y)\nabla^{\perp}_{X}L - g(X, Y)\nabla^{\perp}_{JX}L = 0, \quad (4.10)$$

which contradicts (4.9). Thus the proof is complete.

From this theorem we obtain

COROLLARY 4.1. *There exist no totally geodesic proper CR submanifolds of an elliptic or hyperbolic complex space.*

From this corollary the following question arises: Since $\|h\| \neq 0$ for any proper CR submanifold of an elliptic or hyperbolic complex space, then how far from zero is $\|h\|$? Thus, instead of pinching theorems for CR submanifolds of elliptic or hyperbolic complex spaces, one should consider theorems concerning different evaluations of $\|h\|$ for classes of CR submanifolds.

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